HEAT CONDUCTION WITH VARIABLE HEAT TRANSMISSION COEFFICIENTS

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An analytical method is shown for determining the temperature field of a body whose variable thermophysical properties are functions of the space coordinates. The problem is solved for a plate and for a cylinder where the thermal conductivity is an exponential function of one space coordinate.

Exact analytical methods of solving the equations of heat conduction with variable coefficients have been so far developed for only a very limited range of problems [1]. Essentially, solutions have been obtained for the one-dimensional case in a half-space ($0 \le x \le \infty$). Moreover, as a rule, closed solutions are expressed in the form of unwieldy functional relations. For this reason, in engineering thermophysics one prefers approximate methods even when an exact solution to a boundary-value problem is obtainable. We will consider the boundary-value problem of heat conduction in a nonhomogeneous medium Ω with variable thermophysical properties, in the following coordinate notation:

$$c(M)\gamma(M)\frac{\partial T}{\partial t} = \operatorname{div}[\lambda(M)\operatorname{grad} T(M,\ t)] + q(M)T(M,\ t) + W(M,\ t), \tag{1}$$

$$[T(M,\ t)]_{t=0} = f(M),\quad l[T(M,\ t)]_{\Gamma} = \varphi(M',\ t), \tag{2}$$

where M is any given point with coordinates x, y, z ($M \in \Omega$), M' is a point on the boundary Γ, and l is the first- or second-order differential operator in space coordinates.

Let

$$T^*(M,\ s) = \int_0^\infty T(M,\ t)\exp(-st)\,dt,$$

then the boundary-value problem (1), (2) in Laplace transforms becomes

$$\operatorname{div}[\lambda(M)\operatorname{grad} T^*(M,\ s)] + [q(M) - c\gamma s]T^*(M,\ s) - c\gamma f(M) + W^* = 0, \tag{3}$$

$$l[T^*(M,\ s)]_{\Gamma} = \varphi^*(M',\ s). \tag{4}$$

An approximate solution to the boundary-value problem (3), (4) is found in the form of a vector subtending the coordinate functions $\psi_1(M)$, $\psi_2(M)$, ..., $\psi_n(M)$ in some n-dimensional functional space Ψ

$$T_n^*(M,\ s) = \Phi^*(M,\ s) + \sum_{k=1}^{n} a_k^*(s)\psi_k(M). \tag{5}$$

Functions $\psi_k(M)$ are continuous, as are also their first derivatives, they are linearly independent, and they satisfy the homogeneous condition

$$l[\psi_k(M)]_{\Gamma} = 0.$$

The initial vector $\Phi^*(M,\ s)$ is subject to condition (4). The coefficient transforms $a_k^*(s)$ are projections of vector $T_n^*(M,\ s)$ on the coordinate axes and are found from the requirement of an orthogonal divergence

$$\varepsilon_n\left[a_1^*(s),\ a_2^*(s),\ \ldots,\ a_n^*(s),\ M\right] = \operatorname{div}(\lambda\operatorname{grad} T_n^*) + (q - c\gamma s)T_n^* + c\gamma f + W^* \neq 0$$

imposed on all coordinate functions [2]

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$$\iiint\limits_{\Omega} \varepsilon_n \cdot \psi_j(M)\, dM = 0 \quad (j = 1,\ 2,\ \ldots,\ n). \tag{6}$$

Integrating over the space Ω transforms the last system into an algebraic system of linear equations in coefficients $a_1^*(s),\ \ldots,\ a_n^*(s)$. The fundamental determinant of system (6) is a Gramme determinant not equal to zero in any choice of coordinate functions. After calculating the coefficient and performing an inverse transformation to original functions in (5), we arrive at the solution to the original problem.

We let $\Omega\{0 \le x \le l\}$, $c\gamma = \text{const}$, $\lambda = \lambda_0 e^{-mx}$, $W = 0$, and $q = 0$ so that Eq. (1) reduces to

$$c\gamma \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_0 e^{-mx} \frac{\partial T}{\partial x}\right). \tag{7}$$

An approximate solution to this equation, under the uniqueness conditions

$$T(0,\ t) = 0,\ T(l,\ t) = \varphi(t),\ T(x,\ 0) = 0 \tag{8}$$

is found in Laplace transforms as the following family of functions:

$$T_n^*(x,\ s) = \frac{e^{mx} - 1}{e^{ml} - 1}\varphi^*(s) + \sum_{k=1}^{n} a_k^*(s)\left(1 - \frac{x}{l}\right)\left(\frac{x}{l}\right)^k. \tag{9}$$

At constant boundary values ($\varphi(t) = T_c = \text{const}$) this solution, in the first approximation, reduces to

$$\frac{T(x,\ \mathrm{Fo})}{T_c} = \frac{e^{mx} - 1}{e^{ml} - 1} + \left(1 - \frac{x}{l}\right)\frac{x}{l} \exp\left[-B(\alpha)\,\mathrm{Fo}\right], \tag{10}$$

where

$$B(\alpha) = \frac{30\left[(\alpha^2 - 4\alpha + 8) - e^{-\alpha}(\alpha^2 - 4\alpha + 8)\right]}{\alpha^3}, \tag{11}$$

$$\alpha = ml,\ \mathrm{Fo} = \frac{a_0 t}{l^2},\ a_0 = \frac{\lambda_0}{c\gamma}.$$

At the limit, according to l'Hopital's rule, we obtain

$$\lim_{\alpha \to 0} B(\alpha) = 10,\ \lim_{m \to 0} \frac{e^{mx} - 1}{e^{ml} - 1} = \frac{x}{l}.$$

Solution (10) at the limit $m \to 0$ is

$$\frac{T(x,\ \mathrm{Fo})}{T_c} = \frac{x}{l} + \left(1 - \frac{x}{l}\right)\frac{x}{l} \exp(-10\ \mathrm{Fo}). \tag{12}$$

In this way, when $\alpha \to 0$ ($ml \le 0.1$), the approximate solution (10) almost coincides with the well known solution for a plate with constant heat transmission coefficients [2]. Solution (10) at $\mathrm{Fo} \to \infty$ approaches the exact solution corresponding to the steady-state problem [1].

For an infinitely large plate $\Omega\{-R \le x \le R\}$ with symmetrical boundary conditions, problem (1)-(2) reduces to

$$\frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial}{\partial \xi}\left(e^{-\omega|\xi|}\frac{\partial T}{\partial \xi}\right),\ \xi = \frac{x}{R},\ \omega = mR, \tag{13}$$

$$T(\xi,\ 0) = T_0,\ [T(\xi,\ \mathrm{Fo})]_{\xi=\pm 1} = \varphi(\mathrm{Fo}). \tag{14}$$

Without detracting from the generality of this method, we let $\varphi(\mathrm{Fo}) = T_0(1 + \mathrm{Pd}\,\mathrm{Fo})$, ($\mathrm{Pd} = bR^2/T_0 a_0$) and obtain in transforms

$$\frac{d}{d\xi}\left(e^{-\omega\xi}\frac{dT^*}{d\xi}\right) - sT^*(\xi,\ s) + T_0 = 0, \tag{15}$$

$$[T^*(\xi,\ s)]_{\xi=1} = \varphi^*(s) = T_0\left(\frac{1}{s} + \frac{\mathrm{Pd}}{s^2}\right),\ \left(\frac{dT^*}{d\xi}\right)_{\xi=0} = 0. \tag{16}$$

An approximate solution to the boundary-value problem will be sought within the family of linear compositions

$$T_n^*(\xi,\ s) = \varphi^*(s) + a_1^*(s)\left[e^{\omega}\left(1 - \frac{1}{\omega}\right) - \frac{e^{\omega\xi}(\omega\xi - 1)}{\omega}\right] + \sum_{k=2}^{n} a_k^*(s)(1 - \xi^2)\,\xi^{2(k-1)}. \tag{17}$$

The first coordinate function in solution (17) has been selected equal to the solution to the equation

$$\frac{\partial}{\partial\xi}\left(e^{-\omega\xi}\frac{\partial T}{\partial\xi}\right)=T_0\mathrm{Pd},$$

except for the constant coefficient.

The relative excess temperature is, to the first approximation, written in the form

$$\theta(\xi,\ \mathrm{Fo},\ \mathrm{Pd},\ \omega)=\frac{T(\xi,\ \mathrm{Fo})-T_0}{T_0}=\mathrm{Pd}\left\{\mathrm{Fo}-[1-\exp(-A(\omega)\mathrm{Fo})]\left[\frac{e^{\omega}}{\omega}\left(1-\frac{1}{\omega}\right)-\frac{e^{\omega\xi}}{\omega}\left(\xi-\frac{1}{\omega}\right)\right]\right\},\tag{18}$$

where

$$A(\omega)=\frac{4\left[e^{\omega}(\omega^4-2\omega^3+2\omega^2)-2\omega^2\right]}{e^{2\omega}(4\omega^3-14\omega^2+22\omega-11)-16e^{\omega}(\omega-1)-5}.\tag{19}$$

In the quasisteady mode ($\exp(-A(\omega)\mathrm{Fo}_1)\approx 0$), solution (18) yields

$$\frac{\theta(\xi,\ \mathrm{Fo},\ \omega)}{\mathrm{Pd}}=\mathrm{Fo}-\left[\frac{e^{\omega}}{\omega}\left(1-\frac{1}{\omega}\right)-\frac{e^{\omega\xi}}{\omega}\left(\xi-\frac{1}{\omega}\right)\right],\ \mathrm{Fo}\geqslant\mathrm{Fo}_1,\tag{20}$$

which coincides with the exact solution.

After removing the indeterminacy according to l'Hopital's rule, we have

$$\lim_{\omega\to 0}A(\omega)=2{,}5,\quad \lim_{\omega\to 0}\left[\frac{e^{\omega}}{\omega}\left(1-\frac{1}{\omega}\right)-\frac{e^{\omega\xi}}{\omega}\left(\xi-\frac{1}{\omega}\right)\right]=\frac{1}{2}(1-\xi^2).$$

At $\omega\to 0$ expression (18) becomes

$$\theta(\xi,\ \mathrm{Fo},\ \mathrm{Pd},\ 0)=\mathrm{Pd}\left\{\mathrm{Fo}-\frac{1}{2}(1-\xi^2)[1-\exp(-2{,}5\mathrm{Fo})]\right\}.\tag{21}$$

The temperature in thin plates ($\omega = mR \le 0.2$) can be calculated according to formula (21), which happens to be the solution with a constant thermal conductivity [2].

For an infinitely long cylinder $\Omega\{x^2+y^2\le R^2\}$ we let $\lambda=\lambda_0 e^{mr}$, $c\gamma=\mathrm{const}$, and $\rho^2=(r/R)^2=\xi^2+\eta^2\le 1$ so that Eq. (1) can be written as

$$\frac{\partial T}{\partial t}=a_0\left[\frac{\partial}{\partial x}\left(e^{m\sqrt{x^2+y^2}}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{m\sqrt{x^2+y^2}}\frac{\partial T}{\partial y}\right)\right].\tag{22}$$

In cylindrical coordinates and with symmetrical boundary conditions

$$T(\rho,\ 0)=T_0,\ [T(\rho,\ \mathrm{Fo})]_{\rho=1}=\varphi(\mathrm{Fo})\tag{23}$$

Eq. (22) reduces to

$$\frac{\partial T}{\partial\,\mathrm{Fo}}=e^{\omega\rho}\left(\frac{\partial^2 T}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial T}{\partial\rho}+\omega\frac{\partial T}{\partial\rho}\right).\tag{24}$$

We will now assume that

$$\lim_{\mathrm{Fo}\to\infty}\varphi(\mathrm{Fo})=T_c=\mathrm{const},\tag{25}$$

Then steady state prevails in the cylinder and $\partial T/\partial\mathrm{Fo}=0$. Integrating the equation

$$e^{\omega\rho}\left(\frac{\partial^2 T}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial T}{\partial\rho}+\omega\frac{\partial T}{\partial\rho}\right)=0,$$

we obtain

$$\rho\frac{\partial T}{\partial\rho}=Ce^{-\omega\rho}.$$

Since $(\rho(\partial T/\partial\rho))_{\rho=0}=0$, hence $C\equiv 0$. When $(T)_{\rho=1}=T_c=\mathrm{const}$, therefore, then $T(\rho)=T_c$ will be the solution to the equation, i.e., the temperature field becomes uniform under steady-state conditions in a continuous nonhomogeneous cylinder, just as in a homogeneous cylinder. While satisfying the condition (25), we seek the solution to problem (24), (23) in the transform domain:

$$T_n^*(\rho,\ s)=\varphi^*(s)+\sum_{k=1}^{n}a_k^*(s)(1-\rho^2)\rho^{2(k-1)}. \tag{26}$$

We construct the divergence for the equation

$$e^{\omega\rho}\left[\frac{d}{d\rho}\left(\rho\frac{dT^*}{d\rho}\right)+\omega\rho\frac{dT^*}{d\rho}\right]-s\rho T^*(\rho,\ s)+T_0\rho=0 \tag{27}$$

at $T^*(\rho,\ s)=T_1^*(\rho,\ s)$ and we require that the resulting expression be orthogonal in function $\psi_1(\rho)=(1-\rho^2)$, in order to obtain

$$\int_0^1\{2e^{\omega\rho}(2\rho+\omega\rho^2)(1-\rho^2)+s(1-\rho^2)^2\rho\}\,d\rho-[T_0-s\varphi^*(s)]\int_0^1(1-\rho^2)\rho d\rho=0,$$

from where

$$a_1^*(s)=\frac{3}{2}[T_0-s\varphi^*(s)]\cdot[s+A(\omega)]^{-1}, \tag{28}$$

where

$$A(\omega)=\frac{12\,[e^{\omega}(2\omega^3-6\omega^2+12\omega-12)+12]}{\omega^4}. \tag{29}$$

when $\varphi(\mathrm{Fo})=T_C=\mathrm{const}$, the relative excess temperature is, to the first approximation,

$$\theta(\rho,\ \mathrm{Fo},\ \omega)=\frac{T(\rho,\ \mathrm{Fo})-T_c}{T_0-T_c}=\frac{3}{2}(1-\rho^2)\exp[-A(\omega)\,\mathrm{Fo}]. \tag{30}$$

The relative excess temperature inside a cylinder, with an exponential temperature drop across the wall

$$\varphi(\mathrm{Fo})=T_c+(T_0-T_c)\exp(-\mathrm{Pd\,Fo})$$

is, to the first approximation,

$$\theta(\rho,\ \mathrm{Fo},\ \mathrm{Pd},\ \omega)=\frac{T(\rho,\mathrm{Fo})-T_c}{T_0-T_c}=\exp(-\mathrm{Pd\,Fo})+\frac{3}{2}\Big\{\exp[-A(\omega)\,\mathrm{Fo}]$$
$$-\frac{1}{[\mathrm{Pd}-A(\omega)]}[\mathrm{Pd}\exp(-\mathrm{Pd\,Fo})-A(\omega)\exp(-A(\omega)\,\mathrm{Fo})]\Big\}(1-\rho^2). \tag{31}$$

In those cases where condition (25) is not satisfied the convergence of the approximate solution to the system can be improved by selecting the coordinate functions as follows. As the first coordinate function we select, accurately down to the constant coefficient, the solution to Eq. (24) for the quasisteady-state period. For a linear temperature rise at the boundary ($\varphi(\mathrm{Fo})=T_0(1+\mathrm{Pd\,Fo})$), for example, we let $\partial T/\partial\mathrm{Fo}=\mathrm{Pd}\,T_0$ in Eq. (24). For the quasisteady mode, then, the solution to the boundary-value problem (24), (23) is written as

$$T(\rho,\ \mathrm{Fo},\ \omega,\ \mathrm{Pd})=T_0(1+\mathrm{Pd\,Fo})-\frac{\mathrm{Pd}\,T_0}{2}\left[e^{-\omega}\left(\frac{1}{\omega^2}+\frac{1}{\omega}\right)-\left(\frac{\rho}{\omega}+\frac{1}{\omega^2}\right)e^{-\omega\rho}\right]. \tag{32}$$

When the surface temperature rises linearly, therefore, the temperature field inside a cylinder will be sought in the form

$$T_n^*(\rho,\ s,\ \omega)=\varphi^*(s)+a_1^*(s)\left[e^{-\omega}\left(\frac{1}{\omega^2}+\frac{1}{\omega}\right)-\left(\frac{\rho}{\omega}+\frac{1}{\omega^2}\right)e^{-\omega\rho}\right]+\sum_{k=2}^{n}a_k^*(s)(1-\rho^{2k}). \tag{33}$$

For simplicity, we will seek the solution to the first approximation only. The determining system (6) reduces to

$$a_1^*(s)\left[\frac{e^{-\omega}(\omega^3+3\omega^2+6\omega+6)-6}{\omega^4}\right.$$
$$\left.+s\frac{e^{-2\omega}(4\omega^4+20\omega^3+54\omega^2+78\omega+39)-48(\omega+1)e^{-\omega}+9}{8\omega^6}\right]$$
$$=\frac{T_0\mathrm{Pd}}{s}\left[\frac{6-e^{-\omega}(\omega^3+3\omega^2+6\omega+6)}{\omega^4}\right],$$

TABLE 1. Approximate Eigenvalues and Functions

k	$s_k^{(n)}$					exact values
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	
$k=1$	10,5	9,8754	9,8696	9,8696	9,8696	9,8696
$k=2$		50,1246	39,9978	39,4893	39,4784	39,4784
$k=3$			142,6322	94,1187	88,8825	88,8264
$k=4$				324,5255	194,4327	157,9137
$k=5$					515,2573	246,7401
$f_1^{(4)} = 1,9999 - 3,2888\rho^2 + 1,6178\rho^4 - 0,3691\rho^6 + 0,0402\,\rho^8$						
$f_2^{(4)} = 1,9661 - 12,5072\rho^2 + 22,4460\rho^4 - 16,2848\rho^6 + 4,3799\rho^8$						
$f_3^{(4)} = 1,6381 - 19,6264\rho^2 + 60,7957\rho^4 - 68,5992\rho^6 + 25,7918\rho^8$						
$f_4^{(4)} = 1,1642 - 18,9394\rho^2 + 78,3586\rho^4 - 114,8404\rho^6 + 54,2570\rho^8$						

from where

$$a_1^*(s) = -\frac{\mathrm{Pd}\,T_0}{2}\left[\frac{1}{s} - \frac{1}{s+D(\omega)}\right],$$

with

$$D(\omega) = \frac{8\omega^2\left[e^{-\omega}(\omega^3+3\omega^2+6\omega)-6\right]}{e^{-2\omega}(4\omega^4+20\omega^3+54\omega^2+78\omega+39)-48(\omega+1)e^{-\omega}+9}. \tag{34}$$

The relative excess temperature is written as

$$\theta(\rho,\ \mathrm{Fo},\ \mathrm{Pd},\ \omega) = \frac{T(\rho,\ \mathrm{Fo}) - T_0}{T_0} = \mathrm{Pd}\left\{\mathrm{Fo} - \frac{1}{2}\left[1-\exp(-D(\omega)\,\mathrm{Fo})\right]\right.$$
$$\left.\times\left[e^{-\omega}\left(\frac{1}{\omega^2}+\frac{1}{\omega}\right) - \left(\frac{\rho}{\omega}+\frac{1}{\omega^2}\right)e^{-\omega\rho}\right]\right\}. \tag{35}$$

The solution to Eq. (35) at a sufficiently large Fo coincides with formula (32).

We calculate the limits by l'Hopital's rule and obtain

$$\lim_{\omega\to 0} D(\omega) = 6,\quad \lim\left[e^{-\omega}\left(\frac{1+\omega}{\omega^2}\right) - e^{-\omega\rho}\left(\frac{\rho\omega+1)}{\omega^2}\right)\right] = \frac{1}{2}(1-\rho^2).$$

At the limit $\omega \to 0$, Eq. (35) yields the corresponding approximate solution for a cylinder with a constant thermal conductivity ($\lim\limits_{\omega\to 0}\lambda_0 e^{\omega\rho} = \lambda_0$)

$$\theta(\rho,\ \mathrm{Fo},\ \mathrm{Pd}) = \left\{\mathrm{Fo} - \frac{1}{4}(1-\rho^2)\left[1-\exp(-6\mathrm{Fo})\right]\right\}\mathrm{Pd}. \tag{36}$$

In relation (33), when $n \geq 2$, the limits have the following properties:

$$\lim_{s\to 0} s a_1^*(s) = \lim_{\mathrm{Fo}\to\infty} a_1(\mathrm{Fo}) = -\frac{T_0\mathrm{Pd}}{2},\quad \lim_{s\to 0} s a_k^*(s) = \lim_{\mathrm{Fo}\to\infty} a_k(\mathrm{Fo}) = 0\quad (k > 2),$$

i.e., the solution in the successive second, third, and following approximations approaches an asymptote at $\mathrm{Fo} \to \infty$ which coincides with the exact solution for the quasisteady mode.

In order to compare the approximate solution obtained by this method with the well known exact solutions, we will show here calculations of the temperature inside a homogeneous sphere with boundary conditions of the first kind.

The relative excess temperature is, to the first approximation,

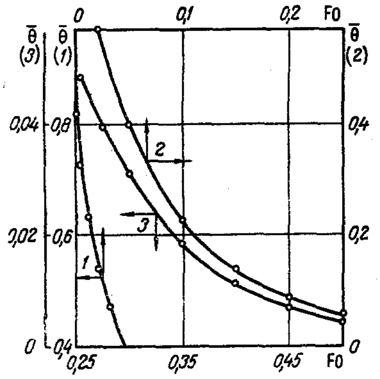

Fig. 1. Mean relative temperature $\bar{\theta}$ as a function of the Fourier number Fo, for a sphere; the dots represent value calculated according to formula (38).

$$\theta_n(\rho,\ \mathrm{Fo}) = \frac{T(\rho,\ \mathrm{Fo}) - T_c}{T_0 - T_c} = \sum_{k=1}^{n} (-1)^{k+1} f_k^{(n)}(\rho) \exp\left(-s_k^{(n)} \mathrm{Fo}\right), \tag{37}$$

where $f_k^{(n)}(\rho)$ are polynomials in even powers 2n. The calculated eigenvalues up to the fifth approximation and the values of functions $f_k^{(n)}(\rho)$ for n = 4 are shown in Table 1. A close agreement is evident between the eigenvalues and the exact values. The polynomial $f_1^{(4)}(\rho)$ almost coincides with the first eigenfunction $2 \sin \pi\rho/\pi\rho$ in the exact solution.

We multiply Eq. (37) by $3\rho^2$ and integrate from 0 to 1, which yields

$$\bar{\theta}_4(\mathrm{Fo}) = 0.6079 \exp(-9.8696\,\mathrm{Fo}) + 0.1523 \exp(-39.4893\,\mathrm{Fo}) + 0.0853 \exp(-94.1187\,\mathrm{Fo}) + 0.0999 \exp(-157.9137\,\mathrm{Fo}). \tag{38}$$

Integrating the first four terms in the exact solution, we have

$$\tilde{\theta}_4(\mathrm{Fo}) = 0.6079 \exp(-9.8696\,\mathrm{Fo}) + 0.1520 \exp(-39.4784\,\mathrm{Fo}) + 0.0675 \exp(-88.8264\,\mathrm{Fo}) + 0.0038 \exp(-157.9137\,\mathrm{Fo}). \tag{39}$$

Obviously, the exact solution is $\tilde{\theta}_\infty(0) = 1$. From (38) and (39) we have $\bar{\theta}_4(0) = 0.9454$ and $\tilde{\theta}_4(0) = 0.8388$. The maximum errors in solutions (38) and (39) are 5.56 and 16.88% respectively, when Fo = 0. On the whole, solution (38) converges better than solution (39).

Solution $\theta_4(\mathrm{Fo})$ is compared with the exact solution graphically in Fig. 1.

When the specific heat and the thermal conductivity both vary according to a symmetrical law

$$c\gamma = c_0\gamma_0(1 + \omega|\xi|)^m, \quad \lambda = \lambda_0(1 + \omega|\xi|)^n, \quad -1 \leqslant \xi = \frac{x}{R} \leqslant 1$$

and when the boundary conditions are symmetrical (14), then the temperature field inside $\Omega\{-1 \le \xi \le 1\}$ is an even function of variable ξ. For this reason, for an infinitely large plate Eq. (1) reduces to

$$(1 + \xi\omega)^m \frac{\partial T}{\partial\,\mathrm{Fo}} = \frac{\partial}{\partial \xi}\left[(1 + \xi\omega)^n \frac{\partial T}{\partial \xi}\right], \quad 0 \leqslant \xi \leqslant 1. \tag{40}$$

Since the derivative of functions $\lambda = \lambda_0(1 + \omega|\xi|)^n$ is discontinuous at the point $\xi = 0$, hence the derivative $\partial T/\partial\xi$ is a discontinuous function at the center of a plate ($\xi = 0$) and $(\partial T/\partial\xi)_{\xi=0} \neq 0$ is possible even with a symmetrical temperature profile across the plate thickness. Consequently, an analysis of the problem on the interval $0 \le \xi \le 1$ brings the singular point $\xi = 0$ to one end of that interval, inside which the temperature together with its derivative remain continuous functions. In this case the condition $(\partial T/\partial\xi)_{\xi=0} = 0$, which is necessary when the thermal conductivity is constant, may not prevail here.

Let us assume that the wall temperature becomes a linear function when Fo is sufficiently large, i.e., that

$$\lim_{\mathrm{Fo}\to\infty} \frac{\varphi(\mathrm{Fo})}{\mathrm{Fo}} = \mathrm{const}, \tag{41}$$

and then the temperature inside a plate will be an asymptotic function during the quasisteady period.

In Eq. (40) we let $\partial T/\partial\,\mathrm{Fo} = T_0\mathrm{Pd}$, and the solution for zero boundary values at point $\xi = 1$ will then be

$$T(\xi,\ \omega,\ m,\ n,\ \mathrm{Pd}) = \frac{\mathrm{Pd}\,T_0}{\omega^2(m+1)}\left\{\left[\frac{(1+\omega\xi)^{m-n+2}}{(m-n+2)} - \frac{(1+\omega\xi)^{1-n}}{1-n}\right] - \left[\frac{(1+\omega)^{m-n+2}}{(m-n+2)} - \frac{(1+\omega)^{1-n}}{(1-n)}\right]\right\}. \tag{42}$$

The solution to problem (40), (41) in Laplace transforms, with condition (41) satisfied, is sought in the form

$$T_n^*(\xi,\ s,\ \mathrm{Pd}) = \varphi^*(s) + a_1^*(s)\left\{\left[\frac{(1+\xi\omega)^{m-n+2}}{(m-n+2)} - \frac{(1+\xi\omega)^{1-n}}{1-n}\right] - \left[\frac{(1+\omega)^{m-n+2}}{(m-n+2)} - \frac{(1+\omega)^{1-n}}{1-n}\right]\right\} + \sum_{k=2}^{n} a_k^*(s)(1-\xi^2)\xi^{2(k-1)}. \tag{43}$$

Having determined the coefficients $a_k^*(s)$ and performed inverse transformations, we find the solution to the original problem. With the given choice of coordinate functions, the following equalities hold true for the limits

$$\lim_{s\to 0} sa_1^*(s) = \lim_{\mathrm{Fo}\to\infty} a_1(\mathrm{Fo}) = \frac{\mathrm{Pd}T_0}{\omega^2(m+1)}, \quad \lim_{s\to 0} sa_k^*(s) = \lim_{\mathrm{Fo}\to\infty} a_k(\mathrm{Fo}) = 0 \quad (k \geqslant 2).$$

If the wall temperature satisfies the condition $\lim_{\mathrm{Fo}\to 0} \varphi(\mathrm{Fo}) = T_0$, then an inverse transformation of (43) will yield for the quasisteady mode ($\mathrm{Fo} \geq \mathrm{Fo}_1$)

$$T(\xi, \mathrm{Fo}, \mathrm{Pd}, \omega) = T_0(1 + \mathrm{Pd}\,\mathrm{Fo}) + \frac{\mathrm{Pd}\,T_0}{\omega^2(m+1)}\left\{\left[\frac{(1+\xi\omega)^{m-n+2}}{m-n+2} - \frac{(1+\xi\omega)^{1-n}}{1-n}\right] - \left[\frac{(1+\omega)^{m-n+2}}{m-n+2} - \frac{(1+\omega)^{1-n}}{1-n}\right]\right\}. \tag{44}$$

We note that

$$\lim \frac{1}{\omega^2}\left\{\left[\frac{(1+\xi\omega)^{m-n+2}}{m-n+2} - \frac{(1+\omega\xi)^{1-n}}{1-n}\right] - \left[\frac{(1+\omega)^{m-n+2}}{m-n+2} - \frac{(1+\omega)^{1-n}}{1-n}\right]\right\} = -\frac{(m+1)}{2}(1-\xi^2)$$

and, to the first approximation, the temperature field (43) in the original domain at $\omega \to 0$ coincides with solution (21).

Since the thermophysical coefficients λ and $c\gamma$ cannot be equal to zero, then the equation of heat conduction (1) does not generate into a line or a point inside region Ω. Its solution will be a continuous function of the coefficients. If coefficients λ and $c\gamma$ are functions of the point M and of the variable ω, and if the conditions

$$\lim_{\omega\to 0} \lambda(M, \omega) = \lambda_0 = \mathrm{const}, \quad \lim_{\omega\to 0} c\gamma = c_0\gamma_0 = \mathrm{const},$$

are satisfied, then the solution to the boundary-value problem (9), (2) is a continuous function of the variable ω. At $\omega \to 0$ the solution approaches the solution of the corresponding problem with constant heat transmission coefficients.

Thus, the described method makes it feasible to analyze the temperature field in one-dimensional and multidimensional cases, in both classical and nonclassical formulation, when the heat transmission coefficients are variable.

With Fo replaced by $X = (1/\mathrm{Pe}) \cdot (z/R)$ in the equation of heat conduction, our method will yield efficient solutions to internal problems of convective heat transfer during turbulent flow through pipes and channels, just as has been shown in [2] for laminar flow.

NOTATION

$T^*(M, s)$	is the Laplace integral transform of the temperature $T(M, t)$;
s	is the Laplace operator;
$M(x, y, z)$	is a given point in region Ω;
Γ	is the boundary of region Ω;
$\mathrm{Pd} = bR^2/a_0T_0$	is the Predvoditelev number for the case of a linearly rising wall temperature;
$\mathrm{Pd} = bR^2/a$	is the Predvoditelev number for the case of an exponentially rising wall temperature.

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